APPLICATION OF THE FINITE ELEMENTS METHOD TO SOLVE THE STATIONARY HEAT-CONDUCTION PROBLEM OF PIECEWISE-INHOMOGENEOUS SYSTEMS

V. S. Sipetov, V. S. Karpilovskii, and O. N. Demchuk UDC 536.24

A finite-element scheme is proposed to solve the stationary heat-conduction problem for piecewise inhomogeneous systems.

The problem of determining temperature fields, their corresponding displacements and stresses in piecewise-inhomogeneous (laminar) bodies is an urgent task. As is noted in [1], a significant simplification in the solution of heat-conduction problems for thin-walled objects can be achieved by introducing the hypothesis of a linear change in temperature over the thickness. Numerical methods must be applied for complex boundary conditions and temperature effects. The finite elements method (FEM) was applied in [2] to the solution of a nonstationary heat-conduction problem for laminar systems, and a number of problems is solved on its basis in [3, 4, 6].

Let us consider a piecewise-inhomogeneous thin-walled shell with an arbitrary quantity of orthotropic layers. The layer contact surfaces are determined by the coordinates a_k and a_{k-1} ($a_k > a_{k-1}$) measured from the x_1Ox_2 coordinate surface to the lower and upper layer boundaries k (k = 1, 2, ..., n). The location of the coordinate surface is chosen arbitrarily, there is ideal thermal contact between layers, and there are no inner heat sources. The temperature distribution over the thickness of the layer packet is taken in the form of a piecewise-linear dependence [5] which is legitimate for thin-walled systems

$$T^{(k)}(x_1,\ x_2,\ z) = T_0(x_1,\ x_2)[1-\psi(z)] + T_n(x_1,\ x_2)\psi(z). \tag{1}$$

Here $\psi(z)$ can be taken in conformity with [5] or in the integral form

$$\psi(z) = \int\limits_{a_0}^{z} (\lambda_3^{(k)})^{-1} dz \Big/ \int\limits_{a_0}^{a_n} (\lambda_3^{(k)})^{-1} dz. \tag{2}$$

Here $\lambda_{(3)}^{k}$ is the heat-conduction coefficient of the k-th layer material in the direction of the normal $x_3 = z$, and a_0, a_n are the z-coordinates of the lower and upper boundaries of the layer packet.

From the variational point of view, the solution of the heat-conduction differential equation under the appropriate boundary conditions is equivalent to seeking the minimum of the functional [3] that takes the following form when the hypothesis (1) is taken into account:

$$U = 0{,}5\int\limits_V \{\lambda_1^{(k)}[T_{0,1}(1-\psi) + T_{n,1}\psi]^2 + \lambda_2^{(k)}[T_{0,2}(1-\psi) + T_{n,2}\psi]^2 + \lambda_3^{(k)}(T_n - T_0)^2\psi_{,3}^2\}\, dV + \int\limits_S [qT + 0{,}5\alpha_s(T - T_\infty)^2]\, dS. \tag{3}$$

Here $\lambda_1^{(k)}$ and $\lambda_2^{(k)}$ are the heat-conduction coefficients in the x_1 and x_2 axis directions, q is the heat flux on part of the surface, α_S is the heat-transfer factor, and T_∞ is the temperature of the environment. Here S is understood to be the shell or plate facial and side surfaces, and T is the corresponding desired or given temperature on these surfaces. Differentiation is denoted by a subscript after a comma.

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TABLE 1. Values of the Temperature Field on the Facial Surfaces of a Three-layered Rectangular Plate

x_2/c_2	x_1/c_1				
	0,1	0,2	0,3	0,4	0,5
0,1	18,69 20,42 43,06 44,04	19,69 20,83 43,54 44,10	19,79 20,84 43,56 44,10	19,79 20,84 43,56 44,10	19,79 20,84 43,56 44,10
0,2	20,07 21,40 43,75 44,18	21,29 21,91 44,28 44,26	21,40 21,91 44,31 44,26	21,43 21,91 44,31 44,26	21,44 21,91 44,31 44,26
0,3	20,25 21,43 43,79 44,19	21,52 21,94 44,32 44,26	21,65 21,95 44,35 44,26	21,67 21,95 44,35 44,26	21,67 21,95 44,35 44,26
0,4	20,27 21,43 43,79 44,19	21,55 21,94 44,33 44,26	21,69 21,95 44,35 44,26	21,70 21,95 44,35 44,26	21,71 21,95 44,35 44,26
0,5	20,28 21,43 43,79 44,19	21,56 21,94 44,33 44,26	21,67 21,95 44,35 44,26	21,71 21,95 44,35 44,26	21,71 21,95 44,35 44,26

To solve the problem under consideration we construct a rectangular finite element (FE) displayed in Fig. 1. We designate two degrees of freedom T_{0i}, T_{ni} in each i-th node (i = 1, 2, 3, 4) of the r-th FE. We approximate the desired temperatures in the FE domain by a known polylinear law

$$T_0 = \sum_{i=1}^{4} T_{0i}\xi_i(x_1, \ x_2); \quad T_n = \sum_{i=1}^{4} T_{ni}\xi_i(x_1, \ x_2). \tag{4}$$

Let us note that along the side of the element the functions T_0 and T_n vary linearly and are determined completely by values given at the nodes belonging to these sides. The conditions for continuity of the desired functions during passage from element to element are thereby conserved. Therefore, the FE has eight degrees of freedom that can be represented in the vector form

$$v_r = \{T_{01}, \ T_{n1}, \ T_{02}, \ \ldots, \ T_{n4}\}_r^T . \tag{5}$$

The system of approximating functions over the FE domain has the following form

$$\Phi_r = \{\Phi_i^r\}_{i=1}^8 = \{\xi_1(1-\psi), \ \xi_1\psi, \ \xi_2(1-\psi), \ \ldots, \ \xi_4\psi\}. \tag{6}$$

The heat-conduction matrix of the FE under consideration can be represented in the form

$$K_r = [K_{ij}^{(0)}]_{i,j=1}^8 + [R_{ij}]_{i,j=1}^8 , \tag{7}$$

where

$$K_{ij}^{(0)} = \int_V [\lambda_1^{(k)}\Phi_{i,1}\Phi_{j,1} + \lambda_2^{(k)}\Phi_{i,2}\Phi_{j,2} + \lambda_3^{(k)}\Phi_{i,3}\Phi_{j,3}]\, dV; \tag{8}$$

$$R_{ij} = \int_S \alpha_s\Phi_i\Phi_j dS. \tag{9}$$

The second term in (7) defined according to (9) permits taking account of heat transfer on the facial and side FE surfaces.

The heat-conduction matrix coefficients are not presented in the present report because of their awkwardness. To illustrate their structure we show one of these coefficients

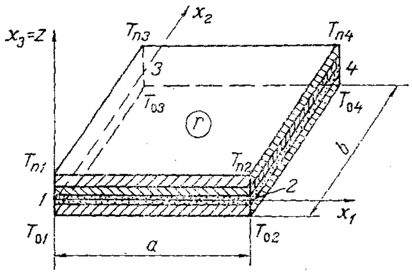

Fig. 1. Rectangular FE of a piecewise-inhomogeneous system.

$$K_{11}^{(0)} + R_{11} = \left[\frac{b}{3a}(B_1 - 2D_1 + P_1) + \frac{a}{3b}(B_2 - 2D_2 + P_2) + \frac{1}{9}\,abP_3\right] + \left[\frac{1}{9}(\alpha_0 ab + 3bH_{13} + 3aH_{12})\right], \quad (10)$$

where a, b are the FE dimensions, and α_0 is heat-transfer factor on the lower surface of the element $z = a_0$. Integral characteristics of the piecewise-inhomogeneous system enter into (10)

$$B_i = \int_{a_0}^{a_n} \lambda_i^{(k)} dz;\ D_i = \int_{a_0}^{a_n} \lambda_i^{(k)} \psi dz; \quad P_i = \int_{a_0}^{a_n} \lambda_i^{(k)} \psi^2 dz;\ P_3 = \int_{a_0}^{a_n} \lambda_3^{(k)} (\psi_{,3})^2 dz;$$
$$H_{1p} = \int_{a_0}^{a_n} (1-\psi)^2 \alpha_{1p}^{(k)} dz;\ i = 1, 2;\ p = 2, 3. \quad (11)$$

Here $\alpha_{12}^{(k)}$ and $\alpha_{13}^{(k)}$ are understood to be the heat-transfer factors over the appropriate side surfaces of the element for the layer k.

The temperature effects distributed over the FE surface must be referred to the nodes to realize the finite-elements method for the computation of piecewise-inhomogeneous systems. The vector of the nodal temperature loads takes the form

$$F_r = \{F_i\}_{i=1}^{8}, \quad (12)$$

where the vector components are determined in conformity with the dependences

$$F_i = \int_S \Phi_i (\alpha_S T_\infty - q)\, dS. \quad (13)$$

It can be considered with sufficient accuracy for the determination of the temperature load vector that the heat-transfer factor, the temperature of the environment, and the heat flux within the limits of the element are constant. Let us show the structure of the temperature load vector component corresponding to the first degree of freedom

$$F_i = \left[\frac{1}{4}(\alpha_0 ab T_\infty^{(0)} + 2b\Lambda_{13} T_\infty^{(13)} + 2a\Lambda_{12} T_\infty^{(12)})\right] - \left[\frac{1}{4}(q_0 ab + 2b\Theta_{13} + 2a\Theta_{12})\right]. \quad (14)$$

Here $T_\infty^{(0)}$ $T_\infty^{(13)}$, $T_\infty^{(12)}$ is the environment temperature at the lower and side (1-3 and 1-2) surfaces of the element, respectively, q_0 is the heat flux on the FE lower surface. The integral characteristics in (14) have the form

$$\Lambda_{1p} = \int_{a_0}^{a_n} (1-\psi)\, \alpha_{1p}^{(k)} dz;\ \Theta_{1p} = \int_{a_0}^{a_n} (1-\psi)\, q_{1p}^{(k)} dz;\ p = 2, 3. \quad (15)$$

Here $q_{13}^{(k)}$ and $q_{12}^{(k)}$ correspond to a given heat flux on the side faces of the element.

Expressions (10) and (14) take account of the thermal flux and convective heat transfer on the facial and side surfaces of the element. However, as is known, they cannot be observed

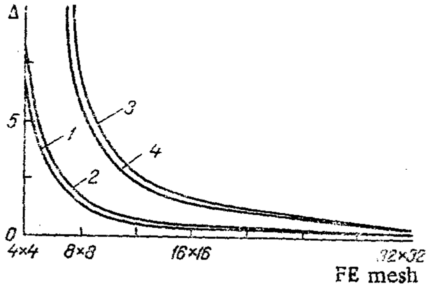

Fig. 2. Graphs of the convergence of the solution for points of the lower (1, 3) and upper (2, 4) surfaces of a five-layered orthotropic plate. Δ, %.

on the identical FE surface. This circumstance is automatically taken into account since one of the components in the relationships (10) and (14) equals zero.

Four different kinds of boundary conditions are associated with the heat-conduction differential equation. The first kind corresponds to a temperature given on a certain part of the boundary (boundary conditions of the first kind)

$$T = T(S). \tag{16}$$

Within the framework of the law (1) taken, this corresponds to given temperatures on the element surfaces. The heat flux intensity is given on the element boundary for boundary conditions of the second kind. This kind of boundary conditions shapes the temperature load vector. The presence of convective heat transfer that takes account of the additional components of (9) in the heat-conduction matrix and a part of the temperature load vector (14) corresponds to boundary conditions of the third kind. Ideal thermal contact conditions (boundary conditions of the fourth kind) are satisfied between the domain individual elements. Therefore, from the FEM viewpoint, the degrees of freedom T_{0i}, T_{ni} can be either given (boundary conditions of the first kind) or desired (boundary conditions of the second, third, and fourth kinds).

Let us note that variation of the functional (3) with respect to independent desired functions T_0 and T_n as well as integration by parts permit a system of heat-conduction differential equations and the appropriate boundary conditions to be obtained. The order of the system is independent of the quantity of layers and equals 4. Since it is difficult to describe this question within limited report, let us state that the equations obtained permit analytic solution of a certain class of problems for laminar orthotropic systems.

Example 1. Numerical Investigation of the Convergence of the Constructed FE. A square plate is considered on whose side surfaces a zero temperature is maintained. Convective heat transfer from the environment occurs on the plate lower surface. The heat transfer factor is $\alpha_0 = 10$ W/(m²·deg), the environment temperature is $T_\infty^{(0)} = 15°$. A heat flux of intensity $q_n = 200$ W/m² is supplied to the upper surface. The plate has a five-layered structure, each layer is a unidirectional epoxy carbon plastic with the following thermophysical characteristics: $\lambda_1 = 14.6$ W/(m·deg), $\lambda_2 = \lambda_3 = 0.93$ W/(m·deg). The layers are oriented alternately at angles 0 and 90° relative to the x_1 axis, the outer layers are oriented at the 0° angle to the x_1 axis. The origin is placed in the left corner of the plate. The layer thicknesses are $h_k = 0.17h, 0.25h, 0.16h, 0.25h, 0.17h$ $(k = 1\text{-}5)$, where $h = 0.1$ m is the total plate thickness. The plate planform dimensions are $c_1 = c_2 = c = 2$ m. The structure under consideration is a thin-walled system $(c/h = 20)$.

An analytic solution is obtained for this problem by expanding the load in a double trigonometric series. We have $T_0 = 29.906°$ and $T_n = 48.457°$ at a point with the coordinates $x_1 = x_2 = 0.5c$ (the point A), and $T_0 = 11.843°$ and $T_n = 19.885°$ at the point with coordinates $x_1 = x_2 = 0.1c$ (point B). A graphical interpretation of the convergence of the numerical solution to the analytic is given in Fig. 2. Curves 1 and 2 describe the convergence of the solution at the point A. Curve 1 corresponds to the temperature on the lower surface and curve 2 on the upper. Curves 3 and 4 describe the convergence of the solution at the point B, respectively. The error Δ, % was determined with respect to the analytic solution as a function of the density of the FE mesh. It is seen from an analysis of the graphs that the element constructed assures convergence of order p^2, i.e., the error of the solution is diminished four times when the FE mesh is compressed twice.

Example 2. Determination of the Stationary Temperature Field in a Three-Layered, Rectangular Plate. The temperature field is determined in a three-layered rectangular (c_1 = 3.5 m, c_2 = 2.9 m) plate to give a foundation to the reliability of the proposed approach. Convective heat transfer from the environment occurs on the lower and upper surfaces of the plate, and a zero temperature is maintained on the side surfaces. The heat transfer factors are α_0 = 8.7 W/(m·deg), α_n = 23 W/(m²·deg), the temperature of the environment is $T^{(0)}$ = 20°, $T^{(n)}$ = 45°. The heat-conduction coefficients are $\lambda_1^{(k)} = \lambda_2^{(k)} = \lambda_3^{(k)} = \lambda$ = (0.52, 0.12, 0.52)/W(m·deg) and the layer thicknesses are h_k = (0.17, 0.1, 0.08) m (k = 1, 2, 3). The origin is placed at the left corner of the plate.

The structure under consideration is the wall panel of a residential building designed for Central Asia. On the basis of the three-dimensional approach eluciated in [7], this problem is solved in [8].

The results of a computation for a quadrant of the plate are presented in Table 1. Values of the temperature at the lower and upper plate surfaces are presented in the first and third rows [8]. Values of the temperature obtained on the basis of the proposed approach are presented in the second and fourth rows for a 40 × 40 FE mesh. As a comparison shows, the values of the temperature found on the basis of the FEM are in good agreement with the results of the more rigorous approach [7, 8].

In conclusion, let us note that the proposed approach can be considered as an extension of the discrete-continual FEM scheme [9] in heat-conduction problems for piecewise-inhomogeneous systems. The crux of this scheme is that discretization is realized only on the system surface; on the thickness each part of it, the finite element, is an inhomogeneous continuum.

In contrast to known finite-element schemes for the solution of heat-conduction problems for laminar systems based on global discretization (over the surface and over the thickness), the proposed approach results in a significant reduction in the number of unknowns and permits examination of thin-walled laminar systems with an arbitrary quantity of layers.

NOTATION

$T_0(x_1, x_2)$, $T_n(x_1, x_2)$, temperature fields on the lower and upper surfaces of a piecewise-inhomogeneous system; $\psi(z)$, temperature field distribution function over the thickness of the layer packet; K_r, heat-conduction matrix of the r-th finite element; F_r, nodal temperature load vector; and q, heat flux.

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